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Abstract. The problem of the alpha-particle S and D state binding energies is formulated in coordinate space and is followed by a variational calculation of the trial function (of the form ($^1S_0 + {}^5D_0$)) contribution to the binding energy of the alpha particle (BE(^{4}He)) using a radial wavefunction of the form

$$\exp(-\lambda \sum_{i>j=2}^{5} r_{ij}^2).$$

The calculation is based on the nuclear forces given by Omojola taking into account the effect of a 'hard core' on BE(^{4}He).

It is shown that, by fixing the value of the variation parameter of $^1S_0$ to the well-established value from the high energy e–^{4}He scattering and by using a repulsive core of suitable radius given by Hamada and Johnston, the excessive binding obtained with two-body central potentials may be reduced to reasonable values. The results obtained are compared with those of other authors.

1. Introduction

Many authors have described variational calculations of the binding energies of the triton and the alpha particle, using the phenomenological potential which satisfied current two-body data (Abraham *et al.* 1955, Irving 1951, 1952, 1953, Clark 1954, Kikuta *et al.* 1956, Blatt and Derrick 1958, Nagata *et al.* 1959, Kanada *et al.* 1963, Irvine 1967, 1968, and Wong 1967). The introduction of a 'hard core' into the two-body potential re-opens interest in further calculations of this kind. Several approaches have been made in the problems of three- and four-particle nuclei and in this paper we describe the direct variational calculations for the alpha particle.

The nuclear potentials of Hamada–Johnston (1962) which are used in this paper contain hard cores in all states of the interacting nucleons. The radii of the cores are assumed to be the same and small ($r_c \sim 0{\cdot}485$ fm).

Brueckner (1958), discussing the hard core, shows that physically acceptable solutions of two-body problems can be obtained if the expression $rV(r)\Phi(r)$ behaves like a Dirac delta function for $0 < r < r_c$, where $\Phi(r)$ is the two-body wavefunction and the hard-core potential V is infinite for $0 < r < r_c$ (r_c is the core radius).

Following the method of Brueckner applied to a two-body scattering problem, the Schrödinger equation involves an integral of the form

$$\int \Psi^*(-1)V(r_{12})\Psi(-1)\, d\tau_{-1}.$$

This integral has to be divided into two regions corresponding to $|r_{12}| < r_c$ and $|r_{12}| > r_c$ if we consider the hard cores of the potential.

However, taking $V(r) \sim (r^2 - r_c^2)w(r)$ in two attractive states and $V(r) \sim -(r^2 + r_c^2)w(r)$ in two nuclear repulsive states, where $w(r)$ is the 'weight' which is proportional to the probability of each configuration (related to binding energy), the calculation greatly simplies and this was adopted here.

Work on the binding energies of ^{3}H and ^{3}He is almost complete and the results obtained will be reported in a future publication.

2. Forms of the nuclear potential

The nucleon–nucleon interaction is known to include both central and non-central forces. According to Rosenfeld (1948) and Okubo and Marshak (1958), the most general form of this interaction with the invariance and symmetry requirements must be a linear combination of the following terms:

(i) The central exchange force

(ii) The linear spin–orbit force

$$S(\boldsymbol{r}_{ij}) = V(\boldsymbol{r}_{ij})\{(\boldsymbol{s}_i+\boldsymbol{s}_j).(\boldsymbol{r}_i-\boldsymbol{r}_j)\times(\boldsymbol{p}_i-\boldsymbol{p}_j)\} \tag{2.1}$$

(iii) The tensor force

$$T(\boldsymbol{r}_{ij}) = V(\boldsymbol{r}_{ij})\{3(\boldsymbol{\sigma}_i\,.\,\boldsymbol{r}_{ij})(\boldsymbol{\sigma}_j\,.\,\boldsymbol{r}_{ij})/r_{ij}^2-(\boldsymbol{\sigma}_i\,.\,\boldsymbol{\sigma}_j)\} \tag{2.2}$$

and

(iv) The quadratic spin–orbit force

$$Q(\boldsymbol{r}_{ij}) = V(\boldsymbol{r}_{ij})\{(\boldsymbol{\sigma}_i\,.\,\boldsymbol{\sigma}_j)L_{ij}^2-Q_{ij}\} \tag{2.3}$$

where

$$\boldsymbol{L}_{ij} = (\boldsymbol{r}_i-\boldsymbol{r}_j)\times(\boldsymbol{p}_i-\boldsymbol{p}_j) \tag{2.4}$$

$$Q_{ij} = \tfrac{1}{2}\{(\boldsymbol{\sigma}_i\,.\,\boldsymbol{L}_{ij})(\boldsymbol{\sigma}_j\,.\,\boldsymbol{L}_{ij})+(\boldsymbol{\sigma}_j\,.\,\boldsymbol{L}_{ij})(\boldsymbol{\sigma}_i\,.\,\boldsymbol{L}_{ij})\}. \tag{2.5}$$

$V(\boldsymbol{r}_{ij})$ is the radial function, $\boldsymbol{s}_i$ is the Pauli spin matrix vector for the ith particle and $\boldsymbol{r}_i$ and $\boldsymbol{p}_i$ are its position and momentum vectors respectively.

Following the convention adopted by Omojola (1970) (to be referred to as II) the interaction can now be written in full as

$$V_{ij} = \sum_{\lambda=1}^{4}\sum_{\nu=1}^{4}\tfrac{1}{4}(w+b_\nu B_{ij}+m_\nu M_{ij}+h_\nu H_{ij})\,{}^{\nu}V_\lambda(\boldsymbol{r}_{ij},\,\boldsymbol{\sigma}_i,\,\boldsymbol{\sigma}_j)+\frac{e_ie_j}{r_{ij}} \tag{2.6}$$

where

$$\begin{aligned}{}^{\nu}V_\lambda(\boldsymbol{r}_{ij},\,\boldsymbol{\sigma}_i,\,\boldsymbol{\sigma}_j) = {} & \sum_{k=1}^{2}{}^{\lambda}U_k^{\nu}\exp(-{}^{\lambda}\mu_k^{\nu}r_{ij}^2)\\ & \times[\delta_{\lambda,1}+\delta_{\lambda,2}\{\tfrac{1}{2}(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\,.\,\boldsymbol{L}_{ij}\}+\delta_{\lambda,3}\{3(\boldsymbol{\sigma}_i\,.\,\boldsymbol{r}_{ij})(\boldsymbol{\sigma}_j\,.\,\boldsymbol{r}_{ij})/r_{ij}^2\\ & -(\boldsymbol{\sigma}_i\,.\,\boldsymbol{\sigma}_j)\}+\delta_{\lambda,4}\{(\boldsymbol{\sigma}_i\,.\,\boldsymbol{\sigma}_j)L_{ij}^2-Q_{ij}\}]\end{aligned} \tag{2.7}$$

e_i is the electronic charge on the ith particle, $\delta_{\lambda\lambda'}$ is the Kronecker delta having its usual meaning, λ represents the kth Gaussian term and takes the symbols C, LS, T and LL for the central, the linear spin–orbit, the tensor and the quadratic spin–orbit force respectively. $\nu = 1, 2, 3$ and 4 to represent the triplet even state, the triplet odd state, the singlet even state and the singlet odd state respectively. The interaction given above is taken to be in the units in which

$$c = \hbar = 1. \tag{2.8}$$

The other parameters given in equations (2.6) and (2.7) are defined in II.

The potential wells $V(\boldsymbol{r}_{ij})$ in this work are taken to be Gaussian forms whose parameters are determined by Omojola (1968) by least-square fits to the Hamada and Johnston potential (Hamada and Johnston 1962). Two Gaussian terms are used for each potential well so that one may represent the short-range contribution and the other the long-range one. The numerical values of U_k and μ_k are given in table 2 (see Appendix 2).

3. The alpha-particle wavefunction

The wavefunction of the ^{4}He system is constructed so that it is antisymmetrical for interchange of the two neutrons and for the two protons respectively. The ground state of ^{4}He is of even parity, its spin is zero and has a total angular momentum $J = 0$. Thus, the possible values of L are 0, 1 and 2. The possible states in operator form have been listed by Gerjuoy and Schwinger (1942), Irving (1953) and Abraham *et al.* (1955). The ground state is a mixture of $^1S_0$, $^3P_0$ and $^5D_0$ states. Only the D state is directly coupled to the S state by the tensor force. Of the six D states listed by Gerjuoy and Schwinger only the principal D state ($^5D_0$) is completely symmetric in the space coordinate and this only is considered in this paper. The principal P state ($^3P_0$) will not be considered here for its contribution is quite negligible for a nuclear force of the type we are going to consider. Also, since the $^3P_0$ state will appear only as a second approximation, it therefore follows that as a first approximation we can neglect the $^3P_0$ state probability.

As shown in II, the wavefunction of the alpha particle can be written in the form

$$\Psi(-1) = \frac{(\Psi_S + c\Psi_D)}{(1+c^2)^{1/2}} \tag{3.1}$$

where

$$\Psi_S = g_S(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4)\chi \tag{3.2}$$

$$\Psi_D = g_D(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \boldsymbol{\rho}_4) W_D \chi \tag{3.3}$$

χ is the spin wavefunction of the singlet state and is defined as

$$\chi(\widetilde{23}, \widetilde{45}) = \tfrac{1}{2}(\alpha_2\beta_3 - \beta_2\alpha_3)(\alpha_4\beta_5 - \beta_4\alpha_5) \equiv \chi(-1) \tag{3.4}$$

2, 3 denote the neutron and 4 and 5 the proton coordinates; g_S and g_D represent the normalized spatial parts of the wavefunctions for the principal $^1S_0$ and $^5D_0$ states respectively. $\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_3$ and $\boldsymbol{\rho}_4$ are the three independent internal space coordinates.

W_D is defined by

$$\begin{aligned} W_D &= \sum_{i>j=2}^{5} r_{ij}^2 s_{ij} \\ &= 3(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}_{23})(\boldsymbol{\sigma}_4 \cdot \boldsymbol{r}_{45}) + 3(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r}_{45})(\boldsymbol{\sigma}_4 \cdot \boldsymbol{r}_{23}) - 2(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_4)(\boldsymbol{r}_{23} \cdot \boldsymbol{r}_{45}) \end{aligned} \tag{3.5}$$

and c^2 determines the amount of the D state in the mixture. We assume that both Ψ_S and Ψ_D are normalized to unity, so that $\Psi(-1)$ is then normalized to unity.

The radial parts Ψ_S and Ψ_D are of the form

$$g_S = N_S \exp\left(-\frac{\alpha}{2} \sum_{i>j=2}^{5} r_{ij}^2\right) \tag{3.6}$$

and

$$g_D = N_D \exp\left(-\frac{\beta}{2} \sum_{i>j=2}^{5} r_{ij}^2\right) \tag{3.7}$$

where N_S and N_D are the normalization constants. The radial parts are assigned different variation parameters, α and β respectively.

The wavefunction of ^{4}He is a function of the relative coordinates of the particles. The following coordinates system of transformation is used:

$$\left.\begin{aligned} \boldsymbol{\rho}_2 &= \boldsymbol{r}_2-\tfrac{1}{3}(\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5) \\ \boldsymbol{\rho}_3 &= \boldsymbol{r}_3-\tfrac{1}{2}(\boldsymbol{r}_4+\boldsymbol{r}_5) \\ \text{and}\quad \boldsymbol{\rho}_4 &= \boldsymbol{r}_4-\boldsymbol{r}_5. \end{aligned}\right\} \tag{3.8}$$

In figure 1 particles 2 and 3 are neutrons and particles 4 and 5 are protons and $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_4$ and $\boldsymbol{r}_5$ are the position vectors of the particles 2, 3, 4 and 5 respectively.

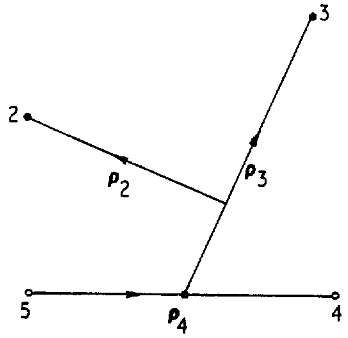

Figure 1

Using equations (3.6), (3.7) and (3.8), equations (3.2) and (3.3) can now be written as

$$\left.\begin{aligned} \Psi_{\mathrm{S}} &= \frac{N_{\mathrm{S}}}{(1+c^2)^{1/2}} \exp\left\{-\frac{\alpha}{2}\left(3\rho_2{}^2+\frac{8}{3}\rho_3{}^2+2\rho_4{}^2\right)\right\} \chi(\widetilde{23},\widetilde{45}) \\ \text{and}\quad \Psi_{\mathrm{D}} &= \frac{cN_{\mathrm{D}}}{(1+c^2)^{1/2}} \exp\left\{-\frac{\beta}{2}\left(3\rho_2{}^2+\frac{8}{3}\rho_3{}^2+2\rho_4{}^2\right)\right\} W_{\mathrm{D}}\chi(\widetilde{23},\widetilde{45}) \end{aligned}\right\} \tag{3.9}$$

where W_{D} is defined by equation (3.5) with $\boldsymbol{r}_{23} = \boldsymbol{\rho}_2-\frac{2}{3}\boldsymbol{\rho}_3$ and $\boldsymbol{r}_{45} = \boldsymbol{\rho}_4$. The normalization coefficients N_{S} and N_{D} are given by

$$\left.\begin{aligned} N_{\mathrm{S}}{}^2 &= \left(\frac{16\alpha^3}{\pi^3}\right)^{3/2} \\ \text{and}\quad N_{\mathrm{D}}{}^2 &= \frac{4\beta^2}{45}\left(\frac{16\beta^3}{\pi^3}\right)^{3/2}. \end{aligned}\right\} \tag{3.10}$$

4. The binding energy calculations

4.1. *The contribution of the energy and potential terms to* BE(^{4}He)

Let H be the complete Hamiltonian of the internal ^{4}He system. Then the total energy inclusive of the D state is given by the following form:

$$E_\alpha = \int \Psi^*(-1)H\Psi(-1)\,\mathrm{d}\tau_{-1} \tag{4.1}$$

where the Hamiltonian is given by

$$H = \left(-\frac{\hbar^2}{2M}\right) \sum_{j=2}^{5} \nabla_j{}^2 + \sum_{i>j=2}^{5} V_{ij}. \tag{4.2}$$

∇^2 is the Laplacian operator acting on the coordinate of the particle whose position vector is $\boldsymbol{r}_i$. $\Psi(-1)$ and V_{ij} have already been defined in this paper.

The Coulomb energy term of ^{4}He is given by

$$E_{\text{Coulomb}} = e^2 \int \Psi^*(-1) \frac{1}{r_{45}} \Psi(-1)\, d\tau_{-1}. \tag{4.3}$$

The contribution of the Coulomb term will be treated as a perturbation. The above expressions have to be minimized in the variational calculation of the binding energy which includes the principal D state.

The forms of the nuclear potential used in this paper are the same as those used in II. The analysis of the contributions of the various interactions will be considered in Appendix 3.

4.2. *The contribution of the 'hard core' to* BE(^{4}He)

We shall now consider the effect of the 'hard core' on BE(^{4}He). Our method of approach will be similar to the method used for a two-body problem by Brueckner (1958). Since we want our potential to vanish at the core radius and also we want it to be infinitely repulsive inside the hard core, we shall assume that

$$\Phi(r)\, r V(r) \;\to \delta(|r| - r_c) \tag{4.4}$$

where r_c is the core radius, $\Phi(r)$ is the spatial part of the wavefunction and r is the distance between two particles. It therefore follows from equation (4.4) that the contribution of the potential term inside the core radius is zero. Hence the only contribution arises from the kinetic energy operator term and this will be considered fully in Appendix 3.

When the effect of the hard core is taken into consideration, the limits of integration become very complicated and thus care must be taken to see that the correct limits of integration are used. To avoid prohibitively complex mathematical analysis the contribution from the D–D terms will be neglected.

The kinetic energy operator is given by

$$\begin{aligned} T_{2345} &= T_{2-345} + T_{3-45} + T_{45} \\ &= \left(-\frac{\hbar^2}{2M}\right) [\tfrac{4}{3}\nabla^2_{2-345} + \tfrac{3}{2}\nabla^2_{3-45} + \tfrac{2}{1}\nabla^2_{45}] \\ &= \left(-\frac{\hbar^2}{2M}\right) [\tfrac{4}{3}\nabla^2 \boldsymbol{\rho}_2 + \tfrac{3}{2}\nabla^2 \boldsymbol{\rho}_3 + \tfrac{2}{1}\nabla^2 \boldsymbol{\rho}_4] \end{aligned} \tag{4.5}$$

on using equation (3.8). Thus

$$\begin{aligned} \langle {}^1S_0{}^* | T_{2345} | {}^1S_0 \rangle = & -2\alpha^2 \frac{\hbar}{M} \frac{N_S{}^2}{1+c^2} \int\int\int \left(-\frac{\partial}{\partial\alpha}\right) \exp\{-\alpha(3\rho_2{}^2 + \tfrac{8}{3}\rho_3{}^2 + 2\rho_4{}^2)\} \\ & \times d\boldsymbol{\rho}_2\, d\boldsymbol{\rho}_3\, d\boldsymbol{\rho}_4 + 18\alpha \frac{\hbar^2}{M} \frac{N_S{}^2}{1+c^2} \\ & \times \int\int\int \exp\{-\alpha(3\rho_2{}^2 + \tfrac{8}{3}\rho_3{}^2 + 2\rho_4{}^2)\}\, d\boldsymbol{\rho}_2\, d\boldsymbol{\rho}_3\, d\boldsymbol{\rho}_4. \end{aligned} \tag{4.6}$$

The evaluation of the above integrals will be dealt with in Appendix 3.

5. Final result for BE(^{4}He)

The total contribution of the kinetic energy, central, linear spin–orbit, tensor, quadratic spin–orbit and the Coulomb forces and the hard core effect to BE(^{4}He) is given in the following form:

$$E_\alpha = \int_{r_c}^{\infty} \Psi^*(-1)H\Psi(-1)\,\mathrm{d}\tau_{-1}$$

$$\equiv \int_0^{\infty} \Psi^*(-1)H\Psi(-1)\,\mathrm{d}\tau_{-1} - \int_0^{r_c} \Psi_S{}^* T_{2345}\Psi_S\,\mathrm{d}\tau_{-1} \tag{5.1}$$

that is

$$\begin{aligned}
(1+c^2)E_\alpha &= \frac{\hbar^2}{2M}(18\alpha+26\beta c^2)_{\mathrm{KE}} \\
&+ \sum_{k=1}^{2}\sum_{v=1}^{4} {}^{\mathrm{C}}U_k^v \Bigg[6(w+m)_v\left(\frac{2\alpha}{2\alpha+{}^{\mathrm{C}}\mu_k^v}\right)^{3/2} \\
&+c^2\Bigg\{\tfrac{5}{2}(w+m+b+h)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{C}}\mu_k^v}\right)^{3/2} + 3(w-m+b-h)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{C}}\mu_k^v}\right)^{5/2} \\
&+\tfrac{1}{2}(w+m+b+h)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{C}}\mu_k^v}\right)^{7/2}\Bigg\}\Bigg] \\
&-3c^2\hbar^2\sum_{k=1}^{2}\sum_{v=3}^{4} {}^{\mathrm{LS}}U_k^v\Bigg\{3(w-m)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{LS}}\mu_k^v}\right)^{5/2} \\
&+(w+m)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{LS}}\mu_k^v}\right)^{7/2}\Bigg\} - \sum_{k=1}^{2}\sum_{v=3}^{4} {}^{\mathrm{T}}U_k^v\Bigg[6\sqrt{5}c(w+m)_v \\
&\times\left(\frac{2\alpha}{\alpha+\beta}\right)\left(\frac{2\beta}{\alpha+\beta}\right)^2\left(\frac{2(\alpha\beta)^{1/2}}{\alpha+\beta+{}^{\mathrm{T}}\mu_k^v}\right)^{5/2}\frac{1}{\alpha+\beta+{}^{\mathrm{T}}\mu_k^v} \\
&+c^2\Bigg\{\tfrac{21}{2}(w-m)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{T}}\mu_k^v}\right)^{5/2} + \tfrac{7}{2}(w+m)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{T}}\mu_k^v}\right)^{7/2}\Bigg\} \\
&\times\frac{1}{2\beta+{}^{\mathrm{T}}\mu_k^v}\Bigg] + \tfrac{2}{3}c^2\hbar^2\sum_{k=1}^{2}\sum_{v=1}^{4} {}^{\mathrm{LL}}U_k^v\Bigg[2(w-m+b-h)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{LL}}\mu_k^v}\right)^{5/2} \\
&-9(w+m+b+h)_v\left(\frac{2\beta}{2\beta+{}^{\mathrm{LL}}\mu_k^v}\right)^{7/2}\Bigg] \\
&+\tfrac{2}{3}e^2\left[3\left(\frac{2\alpha}{\pi}\right)^{1/2}+2c^2\left(\frac{2\beta}{\pi}\right)^{1/2}\right]_{\mathrm{Coulomb}} - (1+c^2)I\Bigg]
\end{aligned} \tag{5.2}$$

where I represents the total contribution of the hard core to BE(^{4}He). Equation (5.2) represents the total contribution from the various interactions—indicated by the symbols attached.

6. Results of the variational calculations of the binding energy of ^{4}He

BE(^{4}He) is calculated from equation (5.2) using Hamada and Johnston potentials expressed in double Gaussians forms whose parameters are determined by Omojola. When equation (5.2) is minimized with respect to α, β and c, BE(^{4}He) is found to be $-58{\cdot}457$ MeV (cf. -120 MeV obtained by Kanada *et al.*). When α is fixed to be $0{\cdot}140\ \mathrm{fm}^{-2}$ (as is well established from high energy e–^{4}He scattering) the BE(^{4}He) is found to be $-26{\cdot}038$ MeV (cf. $-23{\cdot}341$ MeV and -33 MeV obtained by Kanada *et al.* and Nagata *et al.* respectively). Our results in detail are as follows.

Table 1. BE(^{4}He)

Type	State	Coupled state	Contribution to BE(^{4}He) (MeV)
Kinetic energy		SS	51·103
		DD	2·907
Coulomb energy		SS	0·845
		DD	0·016
Hard core (I)		SS	−3·875
Central	$^1V^+$	SS	−47·348
		DD	0·000
	$^1V^-$	SS	0·000
		DD	0·000
	$^3V^+$	SS	−22·815
		DD	−0·775
	$^3V^-$	SS	0·000
		DD	−0·065
Spin–orbit	$^3V^+$	SS	0·000
		DD	−0·014
	$^3V^-$	SS	0·000
		DD	0·963
Tensor	$^3V^+$	SD	−6·824
		DD	0·060
	$^3V^-$	SD	0·000
		DD	−0·189
Quadratic spin–orbit	$^1V^+$	SS	0·000
		DD	0·000
	$^1V^-$	SS	0·000
		DD	0·000
	$^3V^+$	SS	0·000
		DD	0·033
	$^3V^-$	SS	0·000
		DD	0·007
Total (E_α(min))			−26·038

$\alpha = 0{\cdot}140\ \mathrm{fm}^{-2}$, $\beta = 0{\cdot}262\ \mathrm{fm}^{-2}$ and $c = 0{\cdot}145$ which give the minimum value of the BE(^{4}He). $^1V^-$, $^1V^-$, $^3V^+$ and $^3V^-$ represent the singlet even state, singlet odd state, triplet even state and triplet odd state respectively.

7. Conclusions

The BE(^{4}He) is calculated by the Direct Search Method (Hooke and Jeeves 1961 and Kaupe 1963) which is modified by the author to avoid any square root of a negative argument during computation processes. The BE(^{4}He) is found to be

$-26 \cdot 038$ MeV when E_α is minimized with respect to the parameters β and c as compared with the experimental value of $-28 \cdot 2$ MeV. When c is fixed to be $0 \cdot 2$ (assuming that there is exactly 4% D state in the mixture) the BE(^{4}He) is found to be $-57 \cdot 432$ MeV with $\alpha = 0 \cdot 432\ \mathrm{fm}^{-2}$ and $\beta = 0 \cdot 473\ \mathrm{fm}^{-2}$. Sugie *et al.* (1957) obtained E_α(min) as $-17 \cdot 8$ MeV with $\alpha = 0 \cdot 195\ \mathrm{fm}^{-2}$, $\beta = 0 \cdot 307\ \mathrm{fm}^{-2}$ and $c = 0 \cdot 19$ by a parabolic interpolation method.

Irvine took into consideration the effect of a hard core on BE(^{4}He) and obtained the following results without the contribution from the Coulomb force. He obtained $-12 \cdot 80$ MeV using the Reid–Moszkowski–Scott interaction and $-14 \cdot 80$ MeV using the Siemen–Dahlblom interaction. We see that by using the value of the parameter α determined from the high energy e–^{4}He scattering and by using the value of the hard core radius of $0 \cdot 485$ fm we arrive at a reasonable value of the binding energy of ^{4}He as compared with the empirical value and the values obtained by other authors.

It should be pointed out that the inclusion of additional D state in the wavefunction for ^{4}He will definitely increase BE (^{4}He) (Abraham *et al.* 1955) so that a value close to the experimental value of $-28 \cdot 2$ MeV may be obtained. An attempt is being made on this type of problem to see how the binding energy of ^{4}He varies with the core radius. Also an attempt is being made to include the second most important D state in the calculation of this kind. Thus from the above results we conclude that BE(^{4}He) may be calculated using a two-body interaction which accounts for the nucleon–nucleon scattering (up to 300 MeV) and deuteron data.

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Appendix 1

In the calculation of BE(^{4}He) the following formulae are used:

$$k \int \exp\{-\lambda(\boldsymbol{R}-\gamma \boldsymbol{r})^2\}\, \mathrm{d}\boldsymbol{R} = 1$$

$$k \int \boldsymbol{R} \exp\{-\lambda(\boldsymbol{R}-\gamma \boldsymbol{r})^2\}\, \mathrm{d}\boldsymbol{R} = \gamma \boldsymbol{r}$$

$$k \int R^2 \exp\{-\lambda(\boldsymbol{R}-\gamma \boldsymbol{r})^2\}\, \mathrm{d}\boldsymbol{R} = \frac{3}{2\lambda} + \gamma^2 r^2.$$

$$k \int R^4 \exp\{-\lambda(\boldsymbol{R}-\gamma \boldsymbol{r})^2\}\, \mathrm{d}\boldsymbol{R} = \frac{15}{4\lambda^2} + \frac{5}{\lambda}\gamma^2 r^2 + \gamma^4 r^4.$$

$$k \int R^6 \exp\{-\lambda(\boldsymbol{R}-\gamma \boldsymbol{r})^2\}\, \mathrm{d}\boldsymbol{R} = \frac{105}{8\lambda^3} + \frac{105}{4\lambda^2}\gamma^2 r^2 + \frac{21}{2\lambda}\gamma^4 r^4 + \gamma^6 r^6$$

$$k\int (\boldsymbol{R}\times\boldsymbol{A})\,.\,(\boldsymbol{R}\times\boldsymbol{B})\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R} = \gamma^2(\boldsymbol{r}\times\boldsymbol{A})\,.\,(\boldsymbol{r}\times\boldsymbol{B})+\frac{1}{\lambda}(\boldsymbol{A}\,.\,\boldsymbol{B})$$

$$k\int R^2(\boldsymbol{A}\,.\,\boldsymbol{R})(\boldsymbol{B}\,.\,\boldsymbol{R})\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R}$$

$$= \left(\frac{7}{2\lambda}+\gamma^2r^2\right)\gamma^2(\boldsymbol{A}\,.\,\boldsymbol{r})(\boldsymbol{B}\,.\,\boldsymbol{r})+\left(\frac{5}{2\lambda}+\gamma^2r^2\right)\frac{1}{2\lambda}(\boldsymbol{A}\,.\,\boldsymbol{B})$$

$$k\int S_{12}(R^2)\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R} = \gamma^2S_{12}(r^2)$$

$$k\int (\boldsymbol{A}\,.\,\boldsymbol{R})S_{12}(R^2)\exp\{-\lambda(\boldsymbol{R}-\gamma\boldsymbol{r})^2\}\,\mathrm{d}\boldsymbol{R} = \left(\frac{7}{2\lambda}+\gamma^2r^2\right)\gamma^2S_{12},\ \text{etc.}$$

where

$$k = \left(\frac{\lambda}{\pi}\right)^{3/2}$$

and

$$S_{12}(\boldsymbol{A}\,.\,\boldsymbol{B}) = S_{12}(\boldsymbol{B}\,.\,\boldsymbol{A})$$
$$= 3(\boldsymbol{\sigma}_1\,.\,\boldsymbol{A})(\boldsymbol{\sigma}_2\,.\,\boldsymbol{B})-(\boldsymbol{\sigma}_1\,.\,\boldsymbol{\sigma}_2)(\boldsymbol{A}\,.\,\boldsymbol{B}).$$

Appendix 2

The nuclear interaction

The numerical values of ${}^{\lambda}U_k^{\nu}$ and ${}^{\lambda}\mu_k^{\nu}$ are given in table 2 below.

Table 2

λ	ν	U_1	μ_1	U_2	μ_2
Central	1	−57·301	0·781	−227·752	3·091
	2	−19·519	1·047	−45·081	2·860
	3	−150·433	0·961	−1331·301	3·569
	4	−9·768	0·378	2171·754	4·094
Spin–orbit	1	8·720	0·956	38·279	3·351
	2	−220·659	1·826	−1926·808	4·712
Tensor	1	−72·680	0·848	−2230·868	3·149
	2	14·016	0·737	231·447	2·755
Quadratic spin–orbit	1	36·934	1·620	673·374	4·909
	2	11·670	1·964	−2332·307	9·035
	3	5·050	1·387	−61·792	4·380
	4	−129·299	2·032	−4459·021	5·751

${}^{\lambda}U_k^{\nu}$ are given in MeV and ${}^{\lambda}\mu_k^{\nu}$ in fm^{-2}. Other constants used in this work are given below:

$$\frac{M}{\hbar^2} = 0{\cdot}241\times10^{26}\ \mathrm{cm}^{-2}\ \mathrm{MeV}^{-1}$$

and

$$e^2 = 1{\cdot}445\times10^{-13}\ \mathrm{cm\ MeV}.$$

Appendix 3

The contributions of various interactions to the BE(^{4}He) which occur in the text are given below.

3.1 *Kinetic energy*

In the system of coordinates given by equation (3.8), it is easily shown that the operator

$$\left(-\frac{\hbar^2}{2M}\right)\sum_{i>j=2}^{5}\nabla_j^2 = \left(-\frac{\hbar^2}{2M}\right)\left(\tfrac{4}{3}\nabla^2\boldsymbol{\rho}_2+\tfrac{3}{2}\nabla^2\boldsymbol{\rho}_3+\tfrac{2}{1}\nabla^2\boldsymbol{\rho}_4\right)$$

where $\nabla^2\boldsymbol{\rho}_2$, $\nabla^2\boldsymbol{\rho}_3$ and $\nabla^2\boldsymbol{\rho}_4$ act on the coordinates $\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_3$ and $\boldsymbol{\rho}_4$ respectively. Thus contribution of the KE to BE(^{4}He) is given by

$$E_{\mathrm{KE}} = -\left(\frac{\hbar^2}{2M}\right)\int \Psi^*(-1)\left(\tfrac{4}{3}\nabla^2\boldsymbol{\rho}_2+\tfrac{3}{2}\nabla^2\boldsymbol{\rho}_3+\tfrac{2}{1}\nabla^2\boldsymbol{\rho}_4\right)\Psi(-1)\,\mathrm{d}\tau_{-1} \qquad \text{(A3.1)}$$

where

$$\mathrm{d}\tau_{-1} = \mathrm{d}\boldsymbol{\rho}_2\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4. \qquad \text{(A3.2)}$$

The wavefunction given by equation (3.1) is now substituted in the expression (A3.1) for the KE and on using the intregration formulae given in Appendix 1 and the transformation already defined in this paper the above integral is simply evaluated and the result is given by

$$E_{\mathrm{KE}} = \frac{1}{1+c^2}\frac{\hbar^2}{2M}(18\alpha+26\beta c^2). \qquad \text{(A3.3)}$$

3.2. *Coulomb force*

The Coulomb force for the two protons is written in the following form:

$$E_{\mathrm{Coul}} = e^2\int\Psi^*(-1)\,\frac{1}{r_{45}}\Psi(-1)\,\mathrm{d}\boldsymbol{\rho}_2\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4. \qquad \text{(A3.4)}$$

This integral may be evaluated to give

$$E_{\mathrm{Coul}} = \frac{1}{1+c^2}\frac{2e^2}{3}\left\{3\left(\frac{2\alpha}{\pi}\right)^{1/2}+2c^2\left(\frac{2\beta}{\pi}\right)^{1/2}\right\}. \qquad \text{(A3.5)}$$

3.3. *Hard core*

The total contribution of the hard core effect to the BE(^{4}He) is given by

$$I \equiv -2\alpha^2\frac{\hbar^2}{M}\frac{N_{\mathrm{S}}^2}{1+c^2}\int\int\int\left(-\frac{\partial}{\partial\alpha}\right)\exp\{-\alpha(3\rho_2^2+\tfrac{8}{3}\rho_3^2+2\rho_4^2)\}$$
$$\times\,\mathrm{d}\boldsymbol{\rho}_2\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4+18\alpha\frac{\hbar^2}{M}\frac{N_{\mathrm{S}}^2}{1+c^2}\int\int\int\exp\{-\alpha(3\rho_2^2+\tfrac{8}{3}\rho_3^2+2\rho_4^2)\}$$
$$\times\,\mathrm{d}\boldsymbol{\rho}_2\,\mathrm{d}\boldsymbol{\rho}_3\,\mathrm{d}\boldsymbol{\rho}_4. \qquad \text{(A3.6)}$$

Limits of integration

The limits of integration are:

(i) $0 < |\boldsymbol{\rho}_2| \leqslant r_c$

(ii) $0 < |\boldsymbol{\rho}_3| \leqslant r_c$

(iii) $0 < |\boldsymbol{\rho}_4| \leqslant r_c$

(iv) $|\boldsymbol{\rho}_2-\tfrac{2}{3}\boldsymbol{\rho}_3| \leqslant r_c$

(v) $|\boldsymbol{\rho}_2\pm\tfrac{1}{2}\boldsymbol{\rho}_4| \leqslant r_c$

and

(vi) $$|\boldsymbol{\rho}_2+\tfrac{1}{3}\boldsymbol{\rho}_3\pm\tfrac{1}{2}\boldsymbol{\rho}_4| \leqslant r_c. \tag{A3.7}$$

For the angular part of the integration we have the following limits:

(vii) $$\cos\theta_{34} \leqslant \pm\,(r_c^2-\rho_3^2-\rho_4^2)/(\rho_3\rho_4)$$

(viii) $$\cos\theta_{23} \leqslant -(r_c^2-\rho_2^2-\tfrac{4}{9}\rho_3^2)/(\tfrac{4}{3}\rho_2\rho_3)$$

and

(ix) $$\cos\theta_{23} \leqslant (r_c^2-\rho_2^2-\tfrac{1}{9}\rho_3^2-\tfrac{1}{4}\rho_4^2)/(\tfrac{2}{3}\rho_2\rho_3). \tag{A3.8}$$

The integration with respect to $\boldsymbol{\rho}_4$ is performed over a solid angle.

Equation (A3.6) can now be written as

$$\begin{aligned} I = {} & -2\alpha^2\frac{\hbar^2}{M}\frac{N_S^2}{1+c^2}16\pi^3\left(-\frac{\partial}{\partial\alpha}\right)\int_{x_1}^{x_2}\sin\theta_{23}\,d\theta_{23}\int_{x_3}^{x_4}\sin\theta_{34}\,d\theta_{34} \\ & \times\int_0^{r_c}\int_0^{r_c}\int_0^{r_c}\exp\{-\alpha(3\rho_2^2+\tfrac{8}{3}\rho_3^2+2\rho_4^2)\}\rho_2^2\rho_3^2\rho_4^2\,d\rho_2\,d\rho_3\,d\rho_4 \\ & +18\alpha\frac{\hbar^2}{2M}\frac{N_S^2}{1+c^2}16\pi^3\int_{x_1}^{x_2}\sin\theta_{23}\,d\theta_{23}\int_{x_3}^{x_4}\sin\theta_{34}\,d\theta_{34} \\ & \times\int_0^{r_c}\int_0^{r_c}\int_0^{r_c}\exp\{-\alpha(3\rho_2^2+\tfrac{8}{3}\rho_3^2+2\rho_4^2)\}\rho_2^2\rho_3^2\rho_4^2\,d\rho_2\,d\rho_3\,d\rho_4 \end{aligned} \tag{A3.9}$$

where

$$\begin{aligned} x_1 &\equiv \cos\theta_{23} = \frac{-3}{4\rho_2\rho_3}(r_c^2-\rho_2^2-\tfrac{4}{9}\rho_3^2) \\ x_2 &\equiv \cos\theta_{23} = \frac{3}{2\rho_2\rho_3}(r_c^2-\rho_2^2-\tfrac{1}{9}\rho_3^2-\tfrac{1}{4}\rho_4^2) \\ x_3 &\equiv \cos\theta_{34} = -\frac{1}{\rho_3\rho_4}(r_c^2-\rho_3^2-\rho_4^2) \\ x_4 &\equiv \cos\theta_{34} = \frac{1}{\rho_3\rho_4}(r_c^2-\rho_3^2-\rho_4^2). \end{aligned} \tag{A3.10}$$

It can be shown that equation (A3.9) leads to

$$\begin{aligned} I = {} & \left(\frac{\alpha}{\pi}\right)^{3/2} r_c\frac{\hbar^2}{M}\left(5+\alpha\frac{\partial}{\partial\alpha}\right)\Bigg[\Bigg\{(1-\exp(-2\alpha r_c^2))(1-\exp(-3\alpha r_c^2))(409-96\alpha r_c^2) \\ & +24\alpha r_c^2[8\{2\alpha r_c^2-(1-\exp(-3\alpha r_c^2))\}-11]+8\alpha r_c^2(5+28\exp(-3\alpha r_c^2)) \\ & \times\exp(-2\alpha r_c^2)\Bigg\}\frac{1}{\sqrt{t}}\int_0^{\sqrt{t}}\exp(-x^2)\,dx+\{(1-\exp(-2\alpha r_c^2))\}\Bigg\{[(112\alpha r_c^2-313) \\ & \times\{1-\exp(-3\alpha r_c^2)\}+72\alpha r_c^2]+224\alpha r_c^2\{1-\exp(-3\alpha r_c^2)\}\exp(-2\alpha r_c^2)\Bigg\} \\ & \times\exp(-t)\Bigg] \end{aligned} \tag{A.311}$$

where

$$t = \tfrac{8}{3}\alpha r_c^2. \tag{A3.12}$$

3.4. *Central force*

The contribution of the central force to the BE(^{4}He) is given by

$$\sum_{\text{spin}}^{\sigma} (-1) \int d\tau_{-1}\, \chi^*(-1)\Psi^*(-1)\Big(\sum (2345) V_{ij}\Big)\chi(-1)\Psi(-1)$$

where

$$V_{ij} = \sum_{\nu=1}^{4} (w + m_\nu M_{ij} + b_\nu B_{ij} + h_\nu H_{ij})\, {}^{\mathrm{C}}V_k^\nu(ij).$$

On substituting $\Psi(-1)$ and carrying out the spin summations and then integrating over complete vectors, we arrive at the following results:

$$E_{\mathrm{C}} = \sum_{k=1}^{2}\sum_{\nu=1}^{4} \frac{{}^{\mathrm{C}}U_k^\nu}{1+c^2}\Bigg[6(w+m)_\nu\left(\frac{2\alpha}{2\alpha + {}^{\mathrm{C}}\mu_k^\nu}\right)^{3/2}$$
$$+ c^2\Bigg\{\tfrac{5}{2}(w+m+b+h)_\nu\left(\frac{2\beta}{2\beta + {}^{\mathrm{C}}\mu_k^\nu}\right)^{3/2}$$
$$+ 3(w-m+b-h)_\nu\left(\frac{2\beta}{2\beta + {}^{\mathrm{C}}\mu_k^\nu}\right)^{5/2} + \tfrac{1}{2}(w+m+b+h)_\nu\left(\frac{2\beta}{2\beta + {}^{\mathrm{C}}\mu_k^\nu}\right)^{5/2}\Bigg\}\Bigg].$$

3.5. *Linear spin–orbit force*

The spin–orbit force gives no contribution in both the S–S and the S–D states to the BE(^{4}He). Only the D–D state contributes to the BE(^{4}He) and this only is considered in the present section.

$$E_{\mathrm{LS}} = \int \langle \Psi_{\mathrm{D}}^* | V_{\mathrm{LS}} | \Psi_{\mathrm{D}} \rangle\, d\tau_{-1}$$

where

$$V_{\mathrm{LS}} = \boldsymbol{L}\,.\,\boldsymbol{S}(w + mM_{ij})_\nu\, {}^{\mathrm{LS}}V_k^\nu(ij)$$

and ${}^{\mathrm{LS}}V_k^\nu(ij)$ has already been defined in this paper.

$$\boldsymbol{L}_i\,.\,\boldsymbol{S}_{ij} = \frac{\hbar}{2}(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)\,.\,(\boldsymbol{r}_i - \boldsymbol{r}_j)\times(\boldsymbol{p}_i - \boldsymbol{p}_j) \qquad \text{and} \qquad \boldsymbol{p}_i = -i\hbar\frac{\partial}{\partial \boldsymbol{r}_i}.$$

The final result of the contribution of the linear spin–orbit force to BE(^{4}He) is given by

$$E_{\mathrm{LS}} = -\frac{3c^2\hbar^2}{1+c^2}\sum_{k=1}^{2}\sum_{\nu=3}^{4} {}^{\mathrm{LS}}U_k^\nu\left\{3(w-m)_\nu\left(\frac{2\beta}{2\beta + {}^{\mathrm{LS}}\mu_k^\nu}\right)^{5/2} + (w+m)_\nu\left(\frac{2\beta}{2\beta + {}^{\mathrm{LS}}\mu_k^\nu}\right)^{7/2}\right\}.$$

3.6. *The tensor force*

The tensor force contributes to the BE(^{4}He) both in the S–D and the D–D states. We shall first of all consider the contribution from the S–D state.

(i) S–D *state*. The total contribution from the S–D and D–S state is given by

$$8\sum_{\nu=3}^{4} (w+m)_\nu \int \langle \Psi_{\mathrm{S}}^* | \mathrm{S}_{24}^{\mathrm{T}} V_{k(24)}^\nu | \Psi_{\mathrm{D}} \rangle\, d\tau_{-1}$$

where

$$\langle \Psi_S^* | S_{24}^{T} V_{k(24)}^{\nu} | \Psi_D \rangle = \frac{cN_S N_D}{1+c^2} \{18(\boldsymbol{r}_{23}\cdot\boldsymbol{r}_{24})(\boldsymbol{r}_{24}\cdot\boldsymbol{r}_{45}) - 6r_{24}{}^2(\boldsymbol{r}_{23}\cdot\boldsymbol{r}_{45})\}$$
$$\times \exp\{-\lambda(\boldsymbol{\rho}_2 - \xi\boldsymbol{r})^2 - \omega(\boldsymbol{\rho}_3 - \epsilon\boldsymbol{\rho}_4)^2 - \eta\rho_4{}^2\}$$

$$\lambda = 3\delta + {}^{T}\mu_k^{\nu} \qquad \xi = \frac{{}^{T}\mu_k^{\nu}}{3(3\delta + {}^{T}\mu_k^{\nu})} \qquad \boldsymbol{r} = \tfrac{3}{2}\boldsymbol{\rho}_4 - \boldsymbol{\rho}_3$$

$$\omega = \frac{\delta(8\delta + 3\,{}^{T}\mu_k^{\nu})}{3\delta + {}^{T}\mu_k^{\nu}} \qquad \epsilon = \frac{{}^{T}\mu_k^{\nu}}{2(8\delta + 3\,{}^{T}\mu_k^{\nu})} \qquad \eta = \frac{8\delta(2\delta + {}^{T}\mu_k^{\nu})}{8\delta + 3\,{}^{T}\mu_k^{\nu}}$$

and

$$\delta = (\alpha+\beta)/2.$$

Thus the above integral leads to

$$-6\sqrt{5}\,\frac{c}{1+c^2}\left(\frac{2\alpha}{\alpha+\beta}\right)\left(\frac{2\beta}{\alpha+\beta}\right)^2 \sum_{k=1}^{2}\sum_{\nu=3}^{4}(w+m)_\nu\,{}^{T}U_k^{\nu}\left(\frac{2(\alpha\beta)^{1/2}}{\alpha+\beta+{}^{T}\mu_k^{\nu}}\right)^{5/2}\frac{1}{\alpha+\beta+{}^{T}\mu_k^{\nu}}.$$

Note that we have no contribution from S_{23} and S_{45}. This is because each contains $\boldsymbol{\sigma}_2$ or $\boldsymbol{\sigma}_4$ only once and the expectation value of $\boldsymbol{\sigma}_2$ or $\boldsymbol{\sigma}_4$ between two similar spin functions is zero.

(ii) D–D *state.* We now consider the contribution of the D–D state to BE(^{4}He). This is given by the following expression:

$$\int \langle \Psi_D^* | {}^{T}V_k^{\nu} S_{ij}(w + mM_{ij})_\nu | \Psi_D \rangle \, d\tau_{-1}$$

$$= 2\int \langle \Psi_D^* | {}^{T}V_k^{\nu} S_{23}(w + mM_{23})_\nu | \Psi_D \rangle \, d\tau_{-1}$$

$$+4\int \langle \Psi_D^* | {}^{T}V_k^{\nu} S_{34}(w + mM_{34})_\nu | \Psi_D \rangle \, d\tau_{-1}$$

$$= -\frac{c^2}{1+c^2}\sum_{k=1}^{2}\sum_{\nu=3}^{4}{}^{T}U_k^{\nu}\,\frac{1}{2\beta+{}^{T}\mu_k^{\nu}}$$
$$\times\left\{\frac{21}{2}(w-m)_\nu\left(\frac{2\beta}{2\beta+{}^{T}\mu_k^{\nu}}\right)^{5/2} + \tfrac{7}{2}(w+m)_\nu\left(\frac{2\beta}{2\beta+{}^{T}\mu_k^{\nu}}\right)^{7/2}\right\}.$$

3.7. *Quadratic spin–orbit force*

The S–S and the S–D contributions of the quadratic spin–orbit force to the BE(^{4}He) can easily be shown to vanish. Thus, only the D–D terms contribute. Consider the following integral:

$$\int \langle \Psi_D^* | {}^{LL}V_k^{\nu} Q_{LL}(ij)(w + mM_{ij} + bB_{ij} + hH_{ij})_\nu | \Psi_D \rangle \, d\tau_{-1}$$

where Q_{LL} has already been defined in this paper. The above integral is easily evaluated by making use of Appendix 1 and the final result is given by:

$$\frac{2}{3}\frac{c^2}{1+c^2}\hbar^2\sum_{k=1}^{2}\sum_{\nu=1}^{4}{}^{LL}U_k^{\nu}\left\{2(w-m+b-h)_\nu\left(\frac{2\beta}{2\beta+{}^{LL}\mu_k^{\nu}}\right)^{5/2}\right.$$
$$\left. -9(w+m+b+h)_\nu\left(\frac{2\beta}{2\beta+{}^{LL}\mu_k^{\nu}}\right)^{7/2}\right\}.$$

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